Journal of Pure and Applied Algebra 35 (1985) 253-267
North-Holland

A FAMILY OF UNSTABLE STEENROD-MODULES WHICH INCLUDES THOSE OF G. CARLSSON*

Donald M. DAVIS
Department of Mathematics, Lehigh University, Bethlehem, PA 18015, USA

Communicated by J. Stasheff
Received 20 September 1983

AMS(MOS) Subj. Class. (1980): 55S10, 18G20, 17A30.

Keywords: Steenrod algebra, homological dimension, nonassociative algebras, real projective space.

1. Introduction

Let $\mathbb{Z}_2[x_0, x_1, \ldots]$ denote a polynomial algebra on generators x_i of degree 1, made into an unstable left module over the mod 2 Steenrod algebra A by $\mathrm{Sq}^1 x_i = x_{i-1}^2$. If x_i is assigned weight 2^i, then the span $X(n)$ of monomials of weight n is an A-submodule whose dual $G(n)$ is the free unstable right A-module on a generator of degree n. Miller [8] considers the direct limit of

$$G(n) \xrightarrow{\mathrm{Sq}^n} G(2n) \xrightarrow{\mathrm{Sq}^{2n}} G(4n) \xrightarrow{\mathrm{Sq}^{4n}} G(8n) \to \cdots; \tag{1.1}$$

the inverse limit of the dual sequence

$$X(n) \leftarrow X(2n) \leftarrow X(4n) \leftarrow X(8n) \leftarrow \cdots \tag{1.2}$$

is the left A-module X_n studied by Carlsson in [3].

The homomorphisms in the sequence (1.2) send $x_0^{i_0} x_1^{i_1} \cdots$ to $x_0^{i_1} x_1^{i_2} \cdots$ if $i_0 = 0$, and to 0 if $i_0 > 0$. Thus, if n is odd, and y_i denotes $\langle x_{i+j} \in X(2^j n) \rangle$, then X_n is spanned by monomials $y_0^{i_0} y_1^{i_1} \cdots$ with $\sum 2^{-j} i_j = \sum_{e \in E(n)} 2^{-e}$, where $E(n) = \{e : 2^e \in n\}$ is the set of exponents in the dyadic expansion of n.

Let $Y = \mathbb{Z}_2[y_0, y_1, \ldots]$, with $\deg(y_i) = 1$, and $\mathrm{Sq}^1 y_i = y_{i+1}^2$. Let weight$(y_i) = 2^{-i}$. Then Y splits as an A-module into $\bigoplus Y_f$, where f ranges over all non-negative dyadic fractions $a/2^i$, and Y_f is spanned by monomials of weight f. The Y_f with $1 \le f < 2$ comprise Carlsson's modules, with $X_n = Y_f$ if n is an odd integer and $f = \sum_{j \in E(n)} 2^{-j}$. Let G_f denote the (right) module dual to Y_f.

* This work was partially supported by a National Science Foundation Research Grant.

0022-4049/85/$3.30 © 1985, Elsevier Science Publishers B.V. (North-Holland)

Miller's observation that $\operatorname{projdim}_{\mathscr{U}}(G_1)=1$, while $\operatorname{projdim}_{\mathscr{U}^{\mathrm{ft}}}(G_1)=0$, was important in his proof of the Sullivan conjecture [8]. Here $\mathscr{U}$ denotes the category of unstable right A-modules, and $\mathscr{U}^{\mathrm{ft}}$ the subcategory of finite-type modules. Our convention is that A decreases degree in right A-modules. We generalize Miller's result to

Theorem 1.3

(a) $$\operatorname{projdim}_{\mathscr{U}}(\Sigma^k G_f)=\begin{cases} k+2i-2 & \text{if } 2^i-2<f<2^{i+1}-2 \text{ with } i\geq 2 \text{ or } k>0,\\ 1 & \text{if } 0<f\leq 2,\ k=0,\\ k+2i-1 & \text{if } f=2^{i+1}-2.\end{cases}$$

(b) $$\operatorname{projdim}_{\mathscr{U}^{\mathrm{ft}}}(\Sigma^k G_f)=\operatorname{projdim}_{\mathscr{U}}(\Sigma^k G_f)-\begin{cases} 0 & \text{if } f=2^{i+1}-2 \text{ for some } i,\\ 1 & \text{otherwise.}\end{cases}$$

Of course, similar statements may be made for the injective dimension of the unstable left A-modules $\Sigma^k Y_f$.

The proof of 1.3 is by induction using exact sequences

$$0\to\Sigma G_{f-1}\to G_f\to G_{2f}\to 0,\quad f\geq 1. \tag{1.4}$$

The induction is initialized by calculating $\operatorname{projdim}(G_f)$ when $1<f\leq 2$, in which case $G_f=\varinjlim_j G(2^j n)$ where $f=\sum_{j\in E(n)}2^{-j}$. Here we need the following. Let $\nu_2(\ )$ denote the exponent of 2.

Theorem 1.5. (a) *Let* $\alpha_l(m)=|\{e\leq l: 2^e\in m\}|$, *and* $d(k,n)$ *be the smallest* l *such that* $2l+1-\alpha_l(n-2)\geq k$. *Then* $\operatorname{projdim}_{\mathscr{U}}(\Sigma^k G(n))\leq d(k,n)$, *with equality if* $n\geq 2^{k-4}-4$.

(b) $\operatorname{projdim}_{\mathscr{U}^{\mathrm{ft}}}(\Sigma^k G(n))=\operatorname{projdim}_{\mathscr{U}}(\Sigma^k G(n))$.

Theorem 1.5 is valid for $n=0$ or 1 with the convention that the binary expansion of a negative number $-r$ is that of 2^L-r with L sufficiently large.

The reader should contrast 1.3 and 1.5 with the category $\mathscr{U}^*$ of unstable left A-modules, where $\operatorname{projdim}_{\mathscr{U}^*}(M)=0$ or ∞ for any module M (cf. [6]), or the category $\mathscr{A}$ of right A-modules (with no instability condition), which is equivalent to the category of left A-modules, satisfying $\operatorname{projdim}_{\mathscr{A}}(M)=0,1$, or ∞ for any module M (cf. [5]). Adams and Margolis [1] showed that for bounded-below left A-modules projdim $=0$ or ∞.

The proofs of 1.3 and 1.5 are presented in Section 2. The author feels that these, together with the novel description of the modules Y_f, are the main results of the paper.

Equality in 1.5 fails in some cases, the first being $\operatorname{projdim}_{\mathscr{U}}(\Sigma^7 G(3))=3<4$. To exemplify 1.5, we offer

$$\operatorname{projdim}_{\mathscr{U}}(\Sigma^4 G(n)) = \begin{cases} 4, & n \equiv 1\ (16), \\ 3, & n \equiv 0, 5, 7, 8, 9, 13, 15\ (16), \\ 2, & \text{otherwise}. \end{cases}$$

$$\operatorname{projdim}_{\mathscr{U}}(\Sigma^5 G(n)) = \begin{cases} 5, & n \equiv 1\ (32), \\ 4, & n \equiv 0, 9, 13, 15, 16, 17, 25, 29, 31\ (32), \\ 2, & n \equiv 2\ (8), \\ 3, & \text{otherwise}. \end{cases}$$

Martin Bendersky has pointed out the following amusing consequence of these calculations. Let $\Omega : \mathscr{U} \to \mathscr{U}$ be the functor defined in [2; p. 103], and Ω^k its k-fold iterate. As in [8; §8], let Ω_t^k denote the tth right derived functor of Ω^k.

Proposition 1.6. (Bendersky). *For any unstable right A-module N, let $(\Omega_t^k N)_n$ denote the (degree n)-part of $\Omega_t^k N$. Then*

$$(\Omega_5^5 N)_n = 0 \quad \textit{unless } n \equiv 1\ (32),$$

$$(\Omega_4^5 N)_n = 0 \quad \textit{unless } n \equiv 0, 1, 9, 13, 15, 16, 17, 25, 29, 31\ (32),$$

$$(\Omega_3^5 N)_n = 0 \quad \textit{if } n \equiv 2\ (8).$$

That these gaps occur in these derived functors when applied to any module N seems rather curious.

The proof of 1.6 is sketched in Section 3. Also proved in Section 3 is the following result, which evolved from interpreting [3; II.12] in our framework.

Theorem 1.7. *If $0 < f_i < 2$ for $1 \le i \le k$, then*

$$\operatorname{projdim}_{\mathscr{U}}\left(\bigotimes_{i=1}^{k} G_{f_i}\right) = \operatorname{injdim}_{\mathscr{U}}\left(\bigotimes_{i=1}^{k} Y_{f_i}\right) = 1,$$

$$\operatorname{projdim}_{\mathscr{U}^{\mathrm{ft}}}\left(\bigotimes_{i=1}^{k} G_{f_i}\right) = \operatorname{injdim}_{\mathscr{U}^{\mathrm{ft}}}\left(\bigotimes_{i=1}^{k} Y_{f_i}\right) = 0.$$

In Section 3 we investigate the splitting of $\tilde{H}_* RP^\infty$ from G_1, which played an important role in [3] and [8]. We obtain analogous homomorphisms for any G_f, but they give splittings only for those G_f isomorphic to G_1, namely $G_{2^{-j}}$ for $j \ge 0$.

The above splitting plus 1.3 imply the '$\le$' part of the following result, which was known to H. Miller.

Theorem 1.8. $\operatorname{projdim}_{\mathscr{U}}(\tilde{H}_* RP^\infty) = 1$.

The final result of Section 3 is an example of a projective object of $\mathscr{U}$ which is not free, in contrast to the situation for most categories of A-modules.

Carlsson utilized a non-associative multiplication on his X_n. This corresponds to the product $y_0^{i_0} y_1^{i_1} \cdots \bullet y_0^{j_0} y_1^{j_1} \cdots = y_1^{i_0+j_0} y_2^{i_1+j_1} \cdots$ in Y, sending $Y_{f_1} \otimes Y_{f_2} \to Y_{(f_1+f_2)/2}$. In

Section 4, we clarify a remark of Miller [9] that Y_1 is the free 'algebra' on one generator, avoiding the counting argument to which he alludes.

2. The modules Y_f and homological dimension

We begin by verifying some of the statements in the early part of Section 1. Then we prove the theorems regarding homological dimension.

Much of [8; 6.17] can be gleaned from the following argument: Let $K_m = K(\mathbb{Z}_2, m)$ and $P(\ )$ and $Q(\ )$ denote the usual primitive and indecomposable functors. Because PH^*K_m is the free unstable left A-module on a generator of degree m ([4], [7], [10]), $\mathrm{Hom}_{\mathscr{U}}(M, QH_*K_m) \approx (M_m)^*$ for any unstable right A-module M. Thus

$$QH_nK_m \approx \mathrm{Hom}_{\mathscr{U}}(G(n)_*, QH_*K_m) \approx (G(n)_m)^*.$$

[11; §8] shows that $QH_*\underline{K}_*$ is $\mathbb{Z}_2[x_0, x_1, \ldots]$ with x_i of bidegree $(1, 2^i)$ and right A-action given by $x_i\mathrm{Sq} = x_i + x_{i-1}$. Since a homomorphism Sq^n in (1.1) sends $\iota_n\theta$ to $i_{2n}\mathrm{Sq}^n\theta$, the dual homomorphism $X(2n) \to X(n)$ sends x to $x\mathrm{Sq}^n$. The Cartan formula then implies $x_0^{j_0}x_1^{j_1}\cdots$ goes to $x_0^{j_1}x_1^{j_2}\cdots$, verifying the first statement of the second paragraph of Section 1.

The short exact sequence

$$0 \to G(n) \to G(2n) \to \Sigma G(2n-1) \to 0, \tag{2.1}$$

which will be used in proving 1.5, follows immediately, since the homomorphism $X(2n) \to X(n)$ is surjective with kernel consisting of all elements divisible by x_0, which is $\Sigma X(2n-1)$.

The short exact sequence 1.4 is dual to

$$0 \to Y_{2f} \xrightarrow{\alpha} Y_f \xrightarrow{\beta} \Sigma Y_{f-1} \to 0,$$

with

$$\alpha(y_0^{i_0}y_1^{i_1}\cdots) = y_1^{i_0}y_2^{i_1}\cdots$$

and

$$\beta(y_0^{j_0}y_1^{j_1}\cdots) = \begin{cases} y_0^{j_0-1}y_1^{j_1}\cdots & \text{if } j_0 > 0, \\ 0 & \text{if } j_0 = 0. \end{cases}$$

We begin by proving the '$\leq$'-part of 1.5. Since $d(0,n) = 0 = \mathrm{projdim}_{\mathscr{U}}(G(n))$, the result is true when $k=0$. The result will be proved by induction on k. Since

$$\Sigma G(2m) \approx G(2m+1) \tag{2.2}$$

and $d(k-1, 2m+1) = d(k, 2m)$, it suffices to prove the result for odd values of n. $\mathrm{Ext}(\ )$ will always refer to $\mathrm{Ext}_{\mathscr{U}}(\ )$. The exact $\mathrm{Ext}(\ , M)$-sequence associated to (2.1) shows that (letting $\dim(\ ) = \mathrm{projdim}_{\mathscr{U}}(\ )$)

$$\dim(\Sigma^k G(2n-1)) \leq \max(\dim(\Sigma^{k-2}G(2n+1)), 1 + \dim(\Sigma^{k-1}G(n))). \tag{2.3}$$

Both $d(k, 2n-1)$ and $1+d(k-1, n)$ are the smallest l such that $2l-\alpha_{l-1}(n-2)\geq k$. Also, $d(k-2, 2n+1)$ is the smallest l such that $2l-\alpha_{l-1}(n-1)\geq k-2$. Since $\alpha_{l-1}(n-1)\leq\alpha_{l-1}(n-2)+1$, $d(k-2, 2n+1)\leq d(k, 2n-1)$. Thus the result follows from (2.3).

Theorem 1.3 is contained in the following result.

Theorem 2.4. (a) *If* $0<f<2$, *then*

$$\mathrm{Ext}^s(G_f, M)=\begin{cases} 0, & s>1, \\ 0, & s=1,\ M \text{ finite type}, \\ \neq 0, & s=1,\ \text{certain } M, \\ \neq 0, & s=0,\ \text{certain } M \text{ of finite type}. \end{cases}$$

(b) *If* $0<f<2$ *and* $k>0$, *then*

$$\mathrm{Ext}^s(\Sigma^k G_f, M)=\begin{cases} 0, & s>k, \\ 0, & s=k,\ M \text{ finite type}, \\ \neq 0, & s=k,\ \text{certain } M, \\ \neq 0, & s=k-1,\ \text{certain } M \text{ of finite type}. \end{cases}$$

(c) *If* $2^i-2<f<2^{i+1}-2$ *with* $i\geq 2$ *and* $k\geq 0$, *then*

$$\mathrm{Ext}^s(\Sigma^k G_f, M)=\begin{cases} 0, & s>k+2i-2, \\ 0, & s=k+2i-2,\ M \text{ finite type}, \\ \neq 0, & s=k+2i-2,\ \text{certain } M, \\ \neq 0, & s=k+2i-3,\ \text{certain } M \text{ of finite type}. \end{cases}$$

(d) *If* $f=2^{i+1}-2$ *with* $i\geq 1$ *and* $k\geq 0$, *then*

$$\mathrm{Ext}^s(\Sigma^k g_f, M)=\begin{cases} =0, & s>k+2i-1, \\ \subseteq M_1, & s=k+2i-1, \\ =M_1, & s=k+2i-1,\ M \text{ finite type}. \end{cases}$$

Proof. (a) and (b): It suffices to prove it when $1\leq f<2$, because there is an isomorphism $Y_{2f}\rightarrow Y_f$ when $0<f<1$ defined by $y_0^{e_0}y_1^{e_1}\cdots\rightarrow y_1^{e_0}y_2^{e_1}\cdots$.

For $1<f<2$, $G_f=\varinjlim_j G(2^j n)$, where the $\varinjlim$ is over a system as in (1.1), and n is an odd integer which depends upon f. By [8; 6.4] there is a short exact sequence

$$0\rightarrow\lim_j{}^1\mathrm{Ext}^{s-1}(\Sigma^k G(2^j n), M)\rightarrow\mathrm{Ext}^s(\Sigma^k G_f, M)$$
$$\rightarrow\varprojlim_j\mathrm{Ext}^s(\Sigma^k G(2^j n), M)\rightarrow 0 \tag{2.5}$$

for any unstable right A-module M.

The 0-part of (a) follows from (2.5), the fact that $\mathrm{Ext}^s(G(l), M)=0$ if $s>0$, and the fact that $\mathrm{Ext}^s(\Sigma^k G(l), M)$ is finite if M has finite type. [This is clearly true when $k=0$, and follows by induction on k, using (2.2) and the Ext-sequence associated to 2.1.] The 0-part of (b) follows similarly, using (2.2) to replace $\Sigma^k G(2^j n)$ by $\Sigma^{k-1}G(2^j n+1)$, and using '$\le$' in 1.5.

The first '$\neq 0$' part of (a) and (b) follows from the folowing result, omitting '+1's' in (a), using (2.5), and (2.2) in (b).

Proposition 2.6. *Let N be an unstable right A-module with a sequence of elements x_j, $j \ge 0$, of degree $2^j n+1$ satisfying $x_{j+1}\mathrm{Sq}^{2^j n} = x_j$; e.g. $N = PH_*K(\mathbb{Z}_2, n+1)$. Let*

$$N(r)_j = \begin{cases} N, & j \le 2^r n+1, \\ 0, & j > 2^r n+1. \end{cases}$$

Let $M = \bigoplus_{r\ge 0} N(r)$. Then $\lim_j^1 \mathrm{Ext}^k(\Sigma^k G(2^j a+1), M) \neq 0$.

Proof. Because $\mathrm{Sq}^{2m}\mathrm{Sq}^{m+1} = \mathrm{Sq}^{2m+1}\mathrm{Sq}^m$, the diagram

$$\begin{tikzcd} 0 \arrow[r] & G(m+1) \arrow[r, "\mathrm{Sq}^{m+1}"] \arrow[d, "\mathrm{Sq}^m"] & G(2m+2) \arrow[r] \arrow[d, "\mathrm{Sq}^{2m}"] & \Sigma G(2m+1) \arrow[r] \arrow[d, "\mathrm{Sq}^{2m}"] & 0 \\ 0 \arrow[r] & G(2m+1) \arrow[r, "\mathrm{Sq}^{2m+1}"] & G(4m+2) \arrow[r] & \Sigma G(4m+1) \arrow[r] & 0 \end{tikzcd} \tag{2.7}$$

commutes. Using the boundary homomorphisms δ in the exact Ext sequences of the rows of (2.7), we obtain a commutative diagram with any M in the second component

$$\begin{tikzcd} \mathrm{Ext}^0(G(n+1)) \arrow[r, "\delta"] & \mathrm{Ext}^1(\Sigma G(2n+1)) \arrow[r, "\delta"] & \cdots \arrow[r, "\delta"] & \mathrm{Ext}^k(\Sigma^k G(2^k n+1)) \\ \mathrm{Ext}^0(G(2n+1)) \arrow[r, "\delta"] \arrow[u, "\mathrm{Sq}_*^n"] & \mathrm{Ext}^1(\Sigma G(4n+1)) \arrow[r, "\delta"] \arrow[u, "\mathrm{Sq}_*^{2n}"] & \cdots \arrow[r, "\delta"] & \mathrm{Ext}^k(\Sigma^k G(2^{k+1} n+1)) \arrow[u, "\mathrm{Sq}_*^{2^k n}"] \\ \mathrm{Ext}^0(G(4n+1)) \arrow[r, "\delta"] \arrow[u, "\mathrm{Sq}_*^{2n}"] & \mathrm{Ext}^1(\Sigma G(8n+1)) \arrow[r, "\delta"] \arrow[u, "\mathrm{Sq}_*^{4n}"] & \cdots \arrow[r, "\delta"] & \mathrm{Ext}^k(\Sigma^k G(2^{k+2} n+1)) \arrow[u, "\mathrm{Sq}_*^{2^{k+1} n}"] \\ \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \end{tikzcd} \tag{2.8}$$

The horizontal arrows δ are isomorphisms, because the surrounding groups in their exact sequences are $\mathrm{Ext}^s(\Sigma^l G(r))$ with $s > l$, which is 0 by the '$\le$'-part of 1.5. Thus it suffices to show that $\lim^1$ of the left column is nonzero, and this column is just

$$M_{n+1} \xleftarrow[p_0]{\mathrm{Sq}^n} M_{2n+1} \xleftarrow[p_1]{\mathrm{Sq}^{2n}} M_{4n+1} \xleftarrow[p_2]{\mathrm{Sq}^{4n}} \cdots$$

M_{2^jn+1} contains elements $y_{j,r}$, $r \geq j$, corresponding to the element x_j in the $N(r)$-component, satisying $p_j(y_{j+1,r}) = y_{j,r}$. The desired $\lim^1$ is

$$\operatorname{coker}\left(\prod_{j\geq 0} M_{2^jn+1} \xrightarrow{f} \prod_{j\geq 0} M_{2^jn+1}\right),$$

where $f(\langle z_j\rangle) = \langle z_j + p_j z_{j+1}\rangle$. If $\langle y_{j,j}\rangle = f(\langle z_j\rangle)$, then for any t

$$z_0 = y_{0,0} + y_{0,1} + \cdots + y_{0,t} + p_0 p_1 \cdots p_t(z_{t+1}).$$

But $p_0 \cdots p_t(z_{t+1})$ has 0 as its first $(t+1)$ components. Thus z_0 must be nonzero in every component. But $z_0 \in \bigoplus_{r\geq 0} N(r)_{n+1}$ must be a finite sum. Thus $\langle y_{j,j}\rangle$ is a nonzero element of $\lim^1$. $\square$

For the final part of (a), let $M = QH_*K(\mathbb{Z}_2, n)$. Then

$$\varprojlim_j \operatorname{Ext}^0(G(2^jn), M) = \varprojlim_j M_{2^jn} \neq 0,$$

and so the result follows from (2.5). For the final part of (b), let $M = PH_*K(\mathbb{Z}_2, n+1)$. Then using diagram 2.8 we have

$$\varprojlim \operatorname{Ext}^{k-1}(\Sigma^{k-1}G(2^jn+1), M) = \varprojlim \operatorname{Ext}^0(G(2^jn+1), M) = \varprojlim M_{2^jn+1} \neq 0.$$

The remainder of the proof of 2.4 is by induction on f. In part (d) the Ext-sequence associated to 1.4 is

$$\begin{aligned}\operatorname{Ext}^{s-1}(\Sigma^k G_{2^i-1}, M) &\to \operatorname{Ext}^{s-1}(\Sigma^{k+1} G_{2^i-2}, M)\\ &\to \operatorname{Ext}^s(\Sigma^k G_{2^{i+1}-2}, M) \to \operatorname{Ext}^s(\Sigma^k G_{2^i-1}, M). \qquad (2.9)\end{aligned}$$

When $i=1$, $\Sigma^{k+1}G_{2^i-2} = S^{k+1}$, where S^m denotes the A-module which is Z_2 in degree m, and 0 elsewhere. Diagram 2.8 with $n=0$ shows

$$\operatorname{Ext}^s(S^{k+1}, M) = \begin{cases} 0, & s \geq k+1, \\ M_1, & s = k. \end{cases} \qquad (2.10)$$

The result when $i=1$ follows from (2.9), using (a) and (b) to evaluate the first and fourth terms, and (2.10) for the second. Part (d) for arbitrary i follows from (2.9) since, by the induction hypothesis, for $s > k+2i-1$ the second and fourth groups are 0, while for $s = k+2i-1$, the second group is $\subseteq M_1$ and $= M_1$ if M finite type, while the first group is 0 if M finite type, and the fourth group is 0.

The proof of (c) divides into two cases. The first is $2^i - 2 < f \leq 2^{i+1} - 4$. The exact sequence is

$$\begin{aligned}&\operatorname{Ext}^{s-1}(\Sigma^k G_{f/2}, M) \to \operatorname{Ext}^{s-1}(\Sigma^{k+1} G_{f/2-1}, M) \to \operatorname{Ext}^s(\Sigma^k G_f, M)\\ &\quad \to \operatorname{Ext}^s(\Sigma^k G_{f/2}, M) \to \operatorname{Ext}^s(\Sigma^{k+1} G_{f/2-1}, M). \qquad (2.11)\end{aligned}$$

By induction on f, the second and fourth groups are 0 if the hypotheses of either of the first two cases of (c) are satisfied. If $s=k+2i-2$, then the second group is nonzero for a certain M, and the first group is clearly 0, unless $f=2^{i+1}-4$ or $k=0$ and $i=2$. In the former case it is 0 provided M is chosen so that $M_1=0$. The latter case follows from the following result.

Lemma 2.12. *If $f=1+2^{-e_1}+\cdots+2^{-e_k}$ with $0<e_1<\cdots<e_k$, $n=1+2^{e_1}+\cdots+2^{e_k}$, and M is formed from $PH_*K(\mathbb{Z}_2,n)$ as in 2.6, then* $\operatorname{Ext}^1(G_f,M)=0$ *and*

$$\operatorname{Ext}^1(\Sigma G_{f-1},M)\neq 0.$$

Proof.

$$\operatorname{Ext}^1(G_f,M)=\lim{}^1\left(M_n \xleftarrow{\mathrm{Sq}^n} M_{2n} \xleftarrow{\mathrm{Sq}^{2n}} M_{4n} \leftarrow\cdots\right)$$

is 0, since cup-square is 0 in $QH^*K(\mathbb{Z}_2,n)$. $\operatorname{Ext}^1(\Sigma G_{f-1},M)$ is

$$\lim{}^1\left(M_{2(n-1)+1} \xleftarrow{\mathrm{Sq}^{2(n-1)}} M_{4(n-1)+1} \xleftarrow{\mathrm{Sq}^{4(n-1)}} M_{8(n-1)+1} \leftarrow\cdots\right),$$

which is nonzero by the proof of 2.6, since $\mathrm{Sq}^{2^k(n-1)}\cdots\mathrm{Sq}^{n-1}\iota_n\neq 0$ in $QH^*K(\mathbb{Z}_2,n)$. □

The fourth part of 2.4(c) follows similarly, using $M=PH_*K(\mathbb{Z}_2,n)$ when $k=0$ and $i=2$.

If $2^{i+1}-4<f<2^{i+1}-2$, the second and fourth groups of (2.11) are 0 where 2.4(c) claims $\operatorname{Ext}^s(\Sigma^k G_f,M)$ to be 0, and are nonzero where $\operatorname{Ext}^s(\Sigma^k G_f,M)$ is claimed to be nonzero. Moreover, the fifth group of (2.11) is nonzero in these latter cases.

This completes the proof of 1.3. The '=' part of 1.5 follows from:

Theorem 2.13. *Let $M(k,n,s)=2+[(n+k-s-2)/2^s]$. Then*

$$\operatorname{Ext}^s(\Sigma^k G(n),S^m)=\begin{cases} 0 & \text{if } s>d(k,n) \text{ or } m<M(k,n,s),\\ \neq 0 & \text{if } s=d(k,n),\ m=M(k,n,s), \text{ and } n\geq 2^{k-4}-4.\end{cases}$$

Proof. The proof is by induction on k. It is true when $k=0$ since $d(0,n)=0$. Using (2.2), it suffices to prove it when n is odd, and so n will be replaced by $2n-1$. The 0-part when $s>d(k,n)$ has already been proved as the '$\leq$'-part of 1.5. The 0-part when $m<M(k,n,s)$ uses the exact sequence (with S^m in the second component):

$$\begin{aligned}&\operatorname{Ext}^{s-1}(\Sigma^{k-2}G(2n+1))\to\operatorname{Ext}^{s-1}(\Sigma^{k-1}G(n))\to\operatorname{Ext}^s(\Sigma^k G(2n-1))\\ &\quad\to\operatorname{Ext}^s(\Sigma^{k-2}G(2n+1))\to\operatorname{Ext}^s(\Sigma^{k-1}G(n)).\end{aligned}\tag{2.14}$$

If $s\leq d(k,2n-1)$ and $m<M(k,2n-1,s)$, then $\operatorname{Ext}^s(\Sigma^{k-2}G(2n+1),S^m)=0$

since $M(k-2, 2n+1, s) = M(k, 2n-1, s)$. Also, $\mathrm{Ext}^{s-1}(\Sigma^{k-1}G(n), S^m) = 0$ since $M(k, 2n-1, s) \leq M(k-1, n, s-1)$.

The '$\neq 0$'-part of 2.13 requires the following technical result.

Lemma 2.15. *If $s \leq d(k, 2n-1)$, then either $M(k, 2n-1, s) = M(k-1, n, s-1)$ or $s = d(k-2, 2n+1)$.*

Proof. Let $\Delta = k - s \geq 0$. The first equation is satisfied unless, for some positive integer a,

$$a\,2^s - 2\Delta + 3 \leq 2n - 1 < a\,2^s - \Delta + 2.$$

The final equality in 2.15 is satisfied unless, for some $\varepsilon > 0$,

$$2(s-\varepsilon) + 1 - \alpha_{s-\varepsilon}(2n-1) \geq s + \Delta - 2.$$

Therefore, at least one of the two conditions is satisfied unless

$$s \geq \Delta - 3 + 2\varepsilon + \alpha_{s-\varepsilon}(N),$$

where N is an odd number between $A\,2^{s-\varepsilon+1} - 2\Delta + 2$ and $A\,2^{s-\varepsilon+1} - \Delta + 2$, i.e. unless

$$s \geq \Delta - 3 + 2\varepsilon + s - \varepsilon + 1 - \alpha(D),$$

where D is an even number between $\Delta - 3$ and $2\Delta - 3$. This is readily checked to be impossible. $\square$

Let $m = M(k, 2n-1, s)$, and consider (2.14) with S^m in the second component.

If $s = d(k-2, 2n+1)$, then the fourth term is nonzero by induction, and the fifth term is 0, since $s = d(k, 2n-1) = d(k-1, n) + 1$. Thus $\mathrm{Ext}^s(\Sigma^k G(2n-1), S^m) \neq 0$ in this case.

If, on the other hand, $M(k, 2n-1, s) = M(k-1, n, s-1)$, then the second term in (2.14) is nonzero. The first term in (2.14) is 0 *unless* $s - 1 = d(k-2, n+1)$ and $M(k-2, 2n+1, s-1) \leq M(k, 2n-1, s)$. The latter is true only if both terms are 0, i.e.

$$2n + k - s - 2 < 2^{s-1}. \tag{2.16}$$

If $k - s \geq 3$, (2.16) is incompatible with the assumption $2n - 1 \geq 2^{k-4} - 4$ unless $2n - 1 = 2^{k-4} - 3$, which will be treated later. If $k - s = 1$, then $s \leq d(k, 2n-1)$ implies $2(k-2) + 1 - \alpha_{k-2}(2n-3) \leq k - 1$, and hence $2n - 3 = A\,2^{k-1} - 1 - 2^j$ for nonnegative $j \leq k - 2$. This is incompatible with (2.16), which in this case says $2n - 1 < 2^{k-2}$. Finally, if $k - s = 2$, $s \leq d(k, 2n-1)$ implies

$$2(k-3) + 1 - \alpha_{k-3}(2n-3) \leq k - 1,$$

and hence, since (2.16) requires $2n < 2^{k-3}$, we have $2n - 3 = 2^{k-3} - 1 - 2^j$ for $2 \leq j \leq k - 4$. Unless $j = 2$, which will be treated later, this is incompatible with $s - 1 = d(k-2, 2n+1)$ since

$$2(k-4)+1-\alpha_{k-4}(2n-1)=k+j-5\geq k-2.$$

The first of the two special cases mentioned in the preceding paragraph is implied by the following result.

Lemma 2.17. *$Ext^{K+1}(\Sigma^{K+4}G(2^K-3),S^2)\neq 0$.*

Proof. In the exact sequence (2.14) in this case, the second term contains at least two $\mathbb{Z}_2$-summands by Lemma 2.18, and the first term is $\mathbb{Z}_2$ by Lemma 2.19. □

Lemma 2.18. $\dim(\mathrm{Ext}^K(\Sigma^{K+3}G(2^{K-1}-1),S^2))\geq 2$.

Proof. In the exact sequence (2.14) which computes this Ext-group, the first and fifth groups are 0 and the second and fourth groups nonzero, all by cases of 2.13 already proved.

Lemma 2.19. $\mathrm{Ext}^K(\Sigma^{K+2}G(2^K-1),S^2)=\mathbb{Z}_2$.

Proof. In the exact sequence 2.14 which computes this Ext-group, the fifth and second groups are $\mathrm{Ext}^{K-\varepsilon}(\Sigma^K G(2^{K-1}+1),S^2)$ for $\varepsilon=0,1$, respectively. These are 0 since $d(K,2^{K-1}+1)=K-1$ and $M(K,2^{K-1}+1,K-1)=3$. The fourth group is $\mathrm{Ext}^K(\Sigma^K G(2^K+1),S^2)$. That this is $\mathbb{Z}_2$ is proved by induction on K, since the exact sequence (2.14) which computes it involves the analogous term using $K-1$ and terms which are 0. □

The proof of 2.13 will be completed by handling the second of the two special cases, i.e. by showing $\mathrm{Ext}^{k-2}(\Sigma^k G(2^{k-3}-3),S^2)\neq 0$. This is an immediate consequence of the following lemma.

Lemma 2.19. *There is a minimal unstable right A-resolution*

$$0\leftarrow\Sigma^k G(2^{k-3}-3)\leftarrow C_0\xleftarrow{\partial_1}\cdots\xleftarrow{\partial_{k-2}}C_{k-2}\leftarrow 0$$

with

$$C_{k-2}=G(2)\oplus D_{k-2},$$

$$C_{k-3}=G(4)\oplus G(3)\oplus D_{k-3},$$

$$C_{k-4}=G(6)\oplus G(5)\oplus G(4)\oplus G(4)'\oplus D_{k-4},$$

and for $2\leq j\leq k-3$, if $\mu=\min(3,2(k-j-3))$, then

$$C_{k-j-3}=\bigoplus_{\varepsilon=0}^{\mu}G(2^j+j+\varepsilon).$$

Also

$$\partial_{k-j-3}(\iota_{2^j+j+\varepsilon})=\iota_{2^{j+1}+j+\varepsilon+1}\mathrm{Sq}^{2^j+1}+\begin{cases}\iota_{2^{j+1}+j+\varepsilon}\mathrm{Sq}^{2^j}, & \varepsilon=1,3,\\ \iota_{2^{j+1}+j+\varepsilon-1}\mathrm{Sq}^{2^j-1}, & \varepsilon=2,\\ 0, & \varepsilon=0\end{cases}\tag{2.20}$$

except the first term is not present in $\partial_1(\iota_{2^{k-4}+k-\delta})$, $\delta=2$ *or* 3, *or in* $\partial_2(\iota_{2^{k-5}+k-2})$, *and we have*

$$\partial_{k-4}(\iota_4')=\iota_8\mathrm{Sq}^4,$$

$$\partial_{k-3}(\iota_3)=\iota_6\mathrm{Sq}^3+\iota_4'\mathrm{Sq}^1,$$

$$\partial_{k-2}(\iota_2)=\iota_4\mathrm{Sq}^2+\iota_3\mathrm{Sq}^1.$$

If g *is any generator of* D_{k-j-3}, *and* $0\le\varepsilon\le 3$, *then the* $\iota_{2^{j+1}+j+\varepsilon}$*-component of* $\partial_{k-j-3}(g)$, *if nonzero, has coefficient* Sq^I *with* $i_1>2^j+1$.

Proof. Assume the minimal resolution $C_0\leftarrow\cdots\xleftarrow{\partial_{k-j-4}}C_{k-j-4}$ is as described. Then the four elements on the RHS of (2.20) are certainly in $\ker(\partial_{k-j-4})$. The only way they could fail to be images of generators of the minimal resolution is by being divisible. If $\iota_{2^{j+1}+j+\varepsilon}\theta+\sum g_l\theta_l$, with g_l generators of D_{k-j-4} and $\theta,\theta_l\in A$ with $|\theta|\le 2^j$, is in $\ker(\partial_{k-j-4})$, then $0=\mathrm{Sq}^{2^{j+1}+1}\theta+\sum\mathrm{Sq}^{I_l}\theta_l$ with $i_{1_l}>2^{j+1}+1$. But this is impossible, since $|\theta|\le 2^j$, so that if θ is in admissible form so is $\mathrm{Sq}^{2^{j+1}+1}\theta$, and this cannot be obtained from Adem relations beginning with Sq^i for $i>2^{j+1}+1$. Thus C_{k-j-3} contains generators mapped as claimed.

We must also show that the images of generators of D_{k-j-3} are as claimed. If $\iota_{2^{j+1}+j+\varepsilon}\mathrm{Sq}^I+\sum g_l\theta_l\in\ker(\partial_{k-j-4})$ with $i_1\le 2^j$ ($i_1=2^j+1$ is excluded because it is hit by generators of C_{k-j-3}) and g_l generators of D_{k-j-4}, then $\mathrm{Sq}^{2^{j+1}+1}\mathrm{Sq}^{i_1}\cdots\mathrm{Sq}^{i_r}=\sum\tau_l\theta_l$, where τ_l is the $\iota_{2^{j+2}+j+1+\varepsilon}$-component of $\partial_{k-j-4}(g_l)$, and hence begins with Sq^i having $i>2^{j+1}+1$. Since the LHS is admissible, and Adem relations only increase the leading Sq^i, no such relation can exist. □

This completes the proof of 2.13. It is optimal since $\mathrm{Ext}^4(\Sigma^7G(3),S^m)=0$ for all m by a minimal resolution calculation.

3. Relationship with $H_*(RP^\infty)$

Recall $H^*RP^\infty\approx\mathbb{Z}_2[x]$. Let P_k denote the ideal generated by x^k. A result central in [8] and [3] is the splitting of P_1 from Y_1, or dually of $\tilde{H}_*RP^\infty$ from G_1. By 1.3 this implies $\mathrm{Ext}^s(M,P_1)=0$ and $\mathrm{Ext}^s(\tilde{H}_*RP^\infty,M)=0$ if $s>0$ and M has finite type. We consider the splitting homomorphisms from our perspective.

Let $Y=\mathbb{Z}_2[y_0,y_1,\ldots]$ and recall the splitting $Y=\bigoplus_f Y_f$, where f ranges over all nonnegative dyadic fractions. The Y_f-component of $(y_0+y_1+\cdots)^k$ is a finite sum, and so can be used to define homomorphisms $\phi_f:P_1\to Y_f$ by $(y_0+y_1+\cdots)^k=\bigoplus_f\phi_f(x^k)$. These homomorphisms are not A-linear when $f\ge 2$ because $\mathrm{Sq}^1(y_0+y_1+\cdots)=y_1^2+y_2^2+\cdots\ne y_0^2+y_1^2+\cdots$. They will be A-linear into the quotient $\bar{Y}_f$, where $\bar{Y}=Y/(y_0^2,y_1^4,y_2^8,\ldots)$ and $\bar{Y}_f$ is the Y_f-component of $\bar{Y}$.

Let I be the ideal in Y generated by $\{y_iy_{j+1}^2+y_jy_{i+1}^2:i,j\ge 0\}$, and let $I_f=I\cap Y_f$. I and I_f are A-submodules. Let [ ] denote the greatest integer function, and, if $f=2^{-e_1}+\cdots+2^{-e_k}$ with $e_1<\cdots<e_k$, define $\alpha(f)=k$.

Proposition 3.1. *$Y_f/I_f \approx P_{[f]+\alpha(f-[f])}$, and the quotient homomorphism $Y_f \to Y_f/I_f$ sends all monomials to the nonzero class of the appropriate degree.*

Proof. Suppose $f-[f]=2^{-e_1}+\cdots+2^{-e_\alpha}$ with $0<e_1<\cdots<e_\alpha$. The bottom class of Y_f is $y_0^{[f]}y_{e_1}\cdots y_{e_\alpha}$. In each degree $\geq [f]+\alpha$, there is a unique monomial $y_0^{e_0}y_1^{e_1}\cdots y_r^{e_r}$ satisfying $0\leq e_i\leq 1$ for $1\leq i<r$ and $e_r=2$. In degree $[f]+\alpha+j$ with $j>0$ this is

$$\begin{cases} y_0^{[f]}y_{e_1}\cdots y_{e_\alpha-1}y_{e_\alpha+1}\cdots y_{e_\alpha+j-1}y_{e_\alpha+j}^2 & \text{if } \alpha>0,\\ y_0^{[f]-1}y_1\cdots y_j^2 & \text{if } \alpha=0. \end{cases}$$

If a monomial is not of this form, it can be written as $M\cdot y_i^2 y_j$ for some $0<i\leq j$ and some monomial M. This monomial is equivalent mod I_f to $M\cdot y_{i-1}y_{j+1}^2$. This procedure can be continued, always increasing the sum of the subscripts, until the desired term, which has maximal sum of subscripts, is obtained.

Thus Y_f/I_f is $\mathbb{Z}_2$ in degrees $\geq [f]+\alpha(f-[f])$, and 0 in lower degrees. It is A-isomorphic to $P_{[f]+\alpha(f-[f])}$ because $\mathrm{Sq}^n(y_0^{e_0}y_1^{e_1}\cdots y_r^{e_r})$ is a sum of $\binom{\sum e_j}{n}$ monomials. $\square$

If $\bar{Y}_f$ is as earlier in this section, then $\bar{Y}_f/\bar{I}_f=0$ if $f\leq 2$, while if $0<f<2$, then $\bar{Y}_f/\bar{I}_f\approx Y_f/I_f\approx P_{[f]+\alpha(f-[f])}$, and the composite

$$P_1 \xrightarrow{\phi_f} \bar{Y}_f \to \bar{Y}_f/\bar{I}_f \approx P_{[f]+\alpha(f-[f])} \tag{3.2}$$

must be 0 unless $f=2^{-j}$ for some $j\geq 0$ because the A-module structure implies that any homomorphism $P_1\to P_k$ is 0 if $k>1$. If $f=2^{-j}$, then (3.2) is nontrivial on the bottom class, and hence is an isomorphism by tightness of the A-module structure. This is merely a restatement of Miller's proof of the splitting of P_1 from Y_1, but shows that although the other Y_f can be fit into this framework, no interesting splittings are derived.

Let $\psi:\bigoplus_{n\geq 1}G(2^n)\to \tilde{H}_*(RP^\infty)$ be the unique homomorphism nonzero on each component, and $K=\ker(\psi)$.

Theorem 3.3. *K is a projective object of $\mathscr{U}$ which is not free.*

Proof. Since $\mathrm{Ext}^s(\tilde{H}_*RP^\infty, M)=0$ for $s>1$ by 1.3 and the splitting dual to 3.2, K is projective by the Ext-sequence of

$$0\to K\to \bigoplus G(2^n)\to \tilde{H}_*RP^\infty\to 0.$$

If K is free, then $K\approx \bigoplus_{\text{certain } n} G(n)$ with generators in 1-1 correspondence with a basis of $K/K\bar{A}$, where $\bar{A}$ is the ideal of elements of positive degree in the Steenrod algebra. In degree ≤ 5, $K/K\bar{A}$ is generated by ι_2, ι_4, and $\iota_8\mathrm{Sq}^3$. But $(\iota_8\mathrm{Sq}^3)\mathrm{Sq}^2=0$, and so $\iota_8\mathrm{Sq}^3$ cannot generate a $G(5)$-summand. $\square$

Theorem 3.3 contrasts with the situation in the category of left unstable A-

modules and the category of bounded-above right A-modules, where all projective objects are free.

Next we prove Theorem 1.8 by showing

Proposition 3.4. $\mathrm{Ext}^1(\tilde{H}_*(RP^\infty), \bigoplus_{n\geq 1} \tilde{H}_*(RP^{2^n})) \neq 0$.

Proof (M. Hopkins). Since

$$\mathrm{Ext}^s\Big(\tilde{H}_*(RP^\infty), \prod_{n\geq 1} \tilde{H}_*(RP^{2^n})\Big) \approx \prod_{n\geq 1} \mathrm{Ext}^s(\tilde{H}_*(RP^\infty), \tilde{H}_*(RP^{2^n})) = 0$$

for $s\geq 0$ (by 1.3 and 3.2 for $s\geq 1$ and an elementary calculation of $\mathrm{Hom}(\tilde{H}_*RP^\infty, \tilde{H}_*RP^{2^n})$ for $s=0$), the exact sequence associated to

$$0 \to \bigoplus \tilde{H}_*RP^{2^n} \to \prod \tilde{H}_*RP^{2^n} \to C \to 0,$$

with $C = \prod \tilde{H}_*RP^{2^n}/\bigoplus \tilde{H}_*RP^{2^n}$, in the second variable show that it suffices to construct a nontrivial homomorphism $\tilde{H}_*RP^\infty \xrightarrow{\phi} C$.

The homomorphism ϕ which in every degree is nontrivial onto every component is A-linear into C because its restriction to $\tilde{H}_*RP^{2^n}$ equals the composite

$$\tilde{H}_*RP^{2^n} \xrightarrow{\Delta_*} \prod_{i\geq n} \tilde{H}_*RP^{2^i} \hookrightarrow C,$$

where Δ_* is induced by the diagonal map and the inclusion is into sequences 0 in the first $n-1$ components, which are ignored mod $\bigoplus \tilde{H}_*RP^{2^n}$. $\square$

Carlsson's Theorem II.12 realizes $\bigotimes^k Y_1$ as the inverse limit of an inverse system Φ_k of all Y_n with $\alpha(n)=k$. A nice inductive proof of this result can be given from our perspective by considering for each m with $\alpha(m)=k-1$ an inverse sequence of Y_{m+2^e} with $2^e>m$, observing that the inverse limit of this sequence is $Y_m \otimes Y_1$, and that these sequences for all m with $\alpha(m)=k-1$ can be amalgamated to form Φ_k, but the inverse limit can be obtained as $(\varprojlim \Phi_{k-1}) \otimes Y_1$.

We refine this argument to realize $\bigotimes^k Y_1$ as an inverse limit of a sequence of $X(n)$'s somewhat similar to (1.2), and from this we deduce

Theorem 3.5.

$$\mathrm{projdim}_{\mathscr{U}}(\textstyle\bigotimes^k G_1) = \mathrm{injdim}_{\mathscr{U}}(\bigotimes^k Y_1) = 1;$$

$$\mathrm{projdim}_{\mathscr{U}^{\mathrm{ft}}}(\textstyle\bigotimes^k G_1) = \mathrm{injdim}_{\mathscr{U}^{\mathrm{ft}}}(\bigotimes^k Y_1) = 0.$$

Proof. Let $X(n,k) = X(\sum_{i=1}^k 2^{in})$, and define $\phi_n : X(n,k) \to X(n-1,k)$ by writing an element of $X(n,k)$ as $x(1)\cdots x(k)$, where $x(i)$ is a monomial of weight 2^{in} and all subscripts of $x(i)$ are $\leq$ those of $x(i+1)$, ($x(i)$ are uniquely determined), and defining $\phi(n)(x(1)\cdots x(k)) = x(1)_1 \cdots x(k)_k$, where $x(i)_i$ is $x(i)$ with all subscripts decreased by i. Define

$$g_j(y_0^{e_0}\cdots y_r^{e_r})=\begin{cases}x_{j-r}^{e_r}\cdots x_{j-0}^{e_0} & \text{if } r\le j,\\ 0 & \text{otherwise,}\end{cases}$$

and $\psi_n:\bigotimes^k Y_1\to X(n,k)$ by

$$\psi_n(w_1\otimes\cdots\otimes w_k)=q_n(w_1)q_{2n}(w_2)\cdots q_{kn}(w_k).$$

This induces an isomorphism $\psi:\bigotimes^k Y_1\to\varprojlim X(n,k)$. [1 – 1: Let $l(w)$ denote the largest subscript in a monomial w of Y. Then $\psi_n(w_1\otimes\cdots\otimes w_k)\neq 0$ if $n\ge\max\{l(w_i)/i:1\le i\le k\}$. Onto: g_j hits all monomials of weight 2^j.] Thus $\bigotimes^k G_1$ is a direct limit of right A-modules $G(n,k)$ dual to $X(n,k)$, and the result follows from 2.5, as it did for G_f with $1\le f<2$. □

Theorem 1.7 is proved by a similar argument.

Finally, we prove 1.6, which is a special case of

Proposition 3.6. *If* $\operatorname{projdim}_{\mathscr{U}}(\Sigma^k G(n))<t$, *then* $(\Omega_t^k N)_n=0$ *for any* N.

Proof. Miller ([8; 8.4]) noted that there is a spectral sequence

$$\operatorname{Ext}^s_{\mathscr{U}}(M,\Omega_t^k N)\Rightarrow\operatorname{Ext}^{s+t}_{\mathscr{U}}(\Sigma^k M,N)$$

for any M and N. If $M=G(n)$, the initial term is 0 unless $s=0$, in which case it is $(\Omega_t^k N)_n$. Thus the spectral sequence collapses to the isomorphism $(\Omega_t^k N)_n\approx\operatorname{Ext}^t_{\mathscr{U}}(\Sigma^k G(n),N)$. But this is 0 by hypothesis. □

4. Carlsson's non-associative multiplication

In this section we relate our Y_f's to the multiplication introduced in [3]. Being a polynomial algebra, Y has a commutative associative product. We introduce on Y the commutative nonassociative product $\cdot$ defined in Section 1.

If x is a symbol in a word (written without extraneous parentheses) in a non-associative algebra, we define depth(x) = (number of left parentheses to left of x) minus (number of right parentheses to left of x). For example, in $(a\cdot(b\cdot c))\cdot(d\cdot e)$ the depths of a,b,c,d,e are $1,2,2,1,1$, respectively.

Definition 4.1. A non-associative algebra is *depth-invariant* if two words are equal whenever they involve the same ordered pairs (symbol, depth).

Depth-invariance is a strong form of commutativity. For example,

$$(a\cdot(b\cdot c))\cdot(d\cdot(e\cdot f))=(a\cdot d)\cdot((b\cdot c)\cdot(e\cdot f))$$

in a depth invariant algebra, but not necessarily in a commutative algebra.

Theorem 4.2. $(Y_1,\cdot)$ *is the free depth-invariant algebra on one generator.*

Proof. Let F denote the free depth-invariant algebra on x. Define $f: F \to Y_1$ by $f(w) = \prod_{x \in w} y_{\mathrm{depth}(x)+1}$, where the product is over all occurrences of x in the word w, and the product refers to the associative product in Y. This is well-defined since the only relations in F are those due to depth-invariance. The definition of $\bullet$ implies that F is a homomorphism since $(\ )\bullet(\ )$ increases depths by 1.

The homomorphism $g: Y_1 \to F$ inverse to f can be determined iteratively by writing a monomial as $y^I y^J$ with $\mathrm{weight}(y^I) = \mathrm{weight}(y^J) = \frac{1}{2}$ and $i \leq j$ whenever y_i occurs in y^I and y_j occurs in y^J. Define $g(y^I y^J) = g(y^{I'}) \bullet g(y^{J'})$, where $y^{I'}$ (resp. $y^{J'}$) is y^I (resp. y^J) with subscripts decreased by 1. This ultimately reduces to $g(y_0) = x$.

Clearly $fg = 1$, and $gf(w)$ is a word with the same (symbol, depth)'s as w, and hence equals w.

Acknowledgements

The author expresses his appreciation to Haynes Miller, Martin Bendersky, Mike Hopkins and Joe Neisendorfer for useful conversations, and to the Institute for Advanced Study, where this work was begun.

References

[1] J.F. Adams and H. Margolis, Module over the Steenrod algebra, Topology 10 (1971) 271–282.

[2] A.K. Bousfield and D.M. Kan, The homotopy spectral sequence of a space with coefficients in a ring, Topology 11 (1972) 79–106.

[3] G. Carlsson, G.B. Segal's Burnside ring conjecture for $(Z/2)^k$, Topology 22 (1983) 83–103.

[4] H. Cartan, Determination des algebres $H_*(\pi, n; Z_2)$ et $H^*(\pi, n; Z_2)$, Sem. H. Cartan 1954/55.

[5] T.Y. Lin and H. Margolis, Homological aspects of modules over the Steenrod algebra, J. Pour Appl. Algebra 9 (1977) 121–129.

[6] W.H. Lin, Unstable modules over the Steenrod algebra, J. Pure Appl. Algebra 9 (1977) 297–299.

[7] A. Liulevicius, The factorization of cyclic reduced powers by secondary cohomology operations, Mem. Amer. Math. Soc. 42 (1962).

[8] H. Miller, The Sullivan conjecture on maps from classifying spaces, Ann. of Math. 120 (1984) 39–87.

[9] H. Miller, The Sullivan fixed-point conjecture and Brown-Gitler spectra, Unpublished manuscript, 1982.

[10] N.E. Steenrod and D.B.A. Epstein, Cohomology Operations (Princeton Univ. Press, Princeton, 1962).

[11] W.S. Wilson, BP Introduction and Sampler, CBMS Regional Conf. Series 48 (1982).